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Inequalities for Rational Functions

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A new hyperbolic area estimate for the level sets of finite Blaschke products is presented.

The following inversion of the usual Sobolev embedding theorem is proved:

$$\|r\|_{W^1_q(\mathbb{D})} \leqslant Cn \, \|r\|_{L^p(\mathbb{D})}, \qquad p > 2, \qquad 1/q = 1/p + 1/2.$$

Here r is a rational function of degree n with poles outside $\mathbb{D}$.

This estimate implies a new inverse theorem for rational approximation of analytic functions with respect to area L^p norm. © 1997 Academic Press

INTRODUCTION

In this paper a new approach to the rational function estimates is presented.

We obtain the following hyperbolic area estimate for the level sets of a Blaschke product B of degree n in the unit disc $\mathbb{D}$:

$$\int_{\{z:\, |B(z)| \leqslant 1/2\}} \frac{dx\, dy}{(1-|z|)^2} \leqslant 32\pi(n+1). \tag{0.1}$$

We construct a pseudoanalytic extension for rational functions r of degree n with poles outside $\mathbb{D}$. A general scheme for the pseudoanalytic extension of inner functions in $\mathbb{D}$ (in Beurling's sense, cf. [7, Sect. 6 of Chap. 2]) was proposed in [6]. In [2, 6] it was applied to singular inner functions and some free interpolation problems. Here we apply it to finite Blaschke products and estimates of rational functions of a given degree.

In particular, this construction, together with (0.1), gives a new simple proof of the main Bernstein-type inequalities [8, 9] for such functions:

$$\|r\|_{B^s_p(\mathbb{T})} \leqslant Cn^s \, \|r\|_{L^q(\mathbb{T})} \tag{0.2}$$

and

$$\|r\|_{B^{1/p}_p(\mathbb{T})} \leqslant Cn^{1/p} \, \|r\|_{BMO(\mathbb{T})}. \tag{0.3}$$

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Here the constant C does not depend on n, $p>0$ and, in the first inequality,

$$q \geqslant 1, \qquad s = \frac{1}{p} - \frac{1}{q} > 0.$$

$B_p^s(\mathbb{T})$ is the usual Besov space on the unit circle $\mathbb{T}$ (see Section 1.3 below).

It was proved [3] that such Bernstein-type inequalities imply the corresponding inverse theorems of the rational approximation theory. In our case they are the well-known theorems of Peller [9] and Pekarskii [8].

The classical inequality (0.2) deals with the L^p norm on $\mathbb{T}$. Is it possible to obtain an analogue of (0.2) in the area metric, that is in the $L^p(\mathbb{D})$ norm instead of the $L^p(\mathbb{T})$ norm? In the last section of the paper we prove such an analogue.

The main Theorem 4 asserts that for a rational function r of degree n with poles outside the unit disc

$$\left(\int_{\mathbb{C}\setminus\mathbb{D}} |f(z)|^p \, dx \, dy\right)^{1/p} \leqslant C_p n^{1/2} \left(\int_{\mathbb{D}} |r(z)|^p \, dx \, dy\right)^{1/p}, \tag{0.4}$$

where $p \geqslant 2$ and

$$f(z) = \frac{r(z)}{B(z)},$$

B being the Blaschke product of degree $n+2$ with the same poles as r itself and two additional poles at infinity.

The estimate (0.4) allows us to obtain an inversion of the Sobolev embedding theorem for rational functions of a given degree. It is well-known that

$$W_q^1(\mathbb{D}) \subset L^p(\mathbb{D}), \qquad 2 < p < \infty, \qquad \frac{1}{q} = \frac{1}{p} + \frac{1}{2}.$$

Theorem 5 of the paper gives the inverse estimate

$$\|r\|_{W_q^1(\mathbb{D})} \leqslant C_p n \, \|r\|_{L^p(\mathbb{D})} \tag{0.5}$$

for any $p > 2$ and any rational function r of degree n with poles outside the disc.

Of course, this estimate is a Bernstein-type inequality in the area norm. Therefore, one can obtain the corresponding inverse approximation theorem from it. This inverse theorem is Theorem 6 of the paper. It asserts that any analytic function $f \in L^p(\mathbb{D})$ satisfying

$$\sum_{n=1}^{\infty} n^{q(1/2-1/p)} R_n(f)^q_{L^p(\mathbb{D})} < \infty$$

belongs to the Besov class $B_q^{1/q-2/p}(\mathbb{T})$. Here $p > 2$, q is any positive number less than $p/2$, and $R_n(f)_{L^p(\mathbb{D})}$ stands for the best approximation of f in $L^p(\mathbb{D})$ by rational functions of degree n with poles outside $\mathbb{D}$.

Section 1 is devoted to some necessary preliminary information.

In Section 2 the hyperbolic area estimate (0.1) is proved.

In Section 3 the pseudoanalytic extension construction is presented and the inequalities (0.2) and (0.3) are proved. The proofs are quite short and use (0.1) essentially.

Section 4 is devoted to area norm estimates. We prove the main Theorem 4 (that is, (0.4)) and Theorem 5 (that is, (0.5)). In order to obtain the corresponding inverse rational approximation result (Theorem 6) we prove another version of the inverse embedding theorem,

$$\|r\|_{B_q^s(\mathbb{T})} \leqslant C n^{1/2+1/q-1/p} \|r\|_{L^p(\mathbb{D})},$$

where, as before, $p > 2$, $0 < q < p/2$, and $s = 1/q - 2/p$.

Afterwards, Theorem 6 on the rational approximation follows immediately [3].

1. PRELIMINARIES

1.1. *Notation*

$z = x + iy$ and $\zeta = \xi + i\eta$ are complex variables.

$\mathbb{D} = \{z: |z| < 1\}$ is the unit disc, $\mathbb{T} = \{z: |z| = 1\}$ is the unit circle.

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

C and c are constants, not necessarily the same throughout a formula.

The inner Luzin cone $\Gamma(e^{i\theta})$ and the outer Luzin cone $\Gamma^*(e^{i\theta})$ are defined for any point $e^{i\theta} \in \mathbb{T}$ as

$$\Gamma(e^{i\theta}) = \{z \in \mathbb{D}: |z - e^{i\theta}| < 2(1 - |z|)\}$$

and

$$\Gamma^*(e^{i\theta}) = \{z: 1/\bar{z} \in \Gamma(e^{i\theta})\}.$$

A Whitney disc is a disc of the form

$$\Delta = \Delta(z) = \{\zeta : |\zeta - z| < \tfrac{1}{2}\rho(z, \mathbb{T})\},$$

where $z \notin \mathbb{T}$ and

$$\rho(z, \mathbb{T}) = |1 - |z||$$

is the distance from z to $\mathbb{T}$.

The cut-off function χ is a $C^\infty(\mathbb{R})$-function such that

$$0 \leqslant \chi \leqslant 1, \qquad \chi(t) = 1, \quad t > 1/2, \qquad \chi(t) = 0, \quad t < 1/4.$$

1.2. *BMO Space*

$\mathrm{BMO} = \mathrm{BMO}(\mathbb{T})$ is the standard space of functions of bounded mean oscillation on the unit circle [7, Chap. 6] endowed with the norm

$$\|f\|_{\mathrm{BMO}} = \|f\|_{L^1(\mathbb{T})} + \sup_I \frac{1}{|I|} \int_I |f - m_I(f)|,$$

where sup is taken over all subarcs $I \subset \mathbb{T}$ and $m_I(f)$ stands for the mean value of f on I.

$\mathrm{BMOA}(\mathbb{C} \backslash \mathbb{D})$ is the space of all analytic functions g of the Hardy class $H^1(\mathbb{C} \backslash \mathbb{D})$ such that $g|_{\mathbb{T}} \in \mathbb{BMO}(\mathbb{T})$.

The main result of the BMO space theory [7, Chap. 6, Corollary 4.5] asserts that for any function $f \in \mathrm{BMO}$ there exists a decomposition

$$f = v + g|_{\mathbb{T}} \qquad \text{almost everywhere on } \mathbb{T}, \tag{1.1}$$

where $g \in \mathrm{BMOA}$ and v is uniformly bounded and such that

$$\|v\|_{L^\infty(\mathbb{T})} \leqslant C\, \|f\|_{\mathrm{BMO}}.$$

1.3. *Besov Spaces*

One can find the standard definition and discussion of the Besov spaces $B_p^s(\mathbb{T})$, $0 < p < \infty$, $s > 0$, in [4, 5, 11]. In particular, in [5] the following description of B_p^s in terms of so-called pseudoanalytic extension was obtained:

Let r be a function, analytic in $\mathbb{D}$ and continuous (say) up to the circle $\mathbb{T}$. Let $\tilde{r}$ be a continuous extension of r to the whole plane such that $\tilde{r} \in C^\infty(\mathbb{C} \backslash \mathbb{D})$. Then for any $p > 0$ and $s > 0$ such that $s > 1/p - 1$

$$\|r\|_{B_p^s(\mathbb{T})} \leqslant C_1 \max_{|z|=2} |\tilde{r}(z)| + C_2 \left(\int_{1<|z|<2} \sigma(z)^p \frac{dx\, dy}{(|z|-1)^{ps+1}} \right)^{1/p}, \tag{1.2}$$

where

$$\sigma(z) = \sup \left\{ (|\zeta|-1) \left| \frac{\partial \tilde{r}}{\partial \bar{\zeta}} \right| : |\zeta - z| < 1/2(|z|-1) \right\}. \tag{1.3}$$

It is well-known (see, e.g., [5]) that $B_p^{1/p}(\mathbb{T}) \subset \mathrm{BMO}(\mathbb{T})$, and so any function $f \in B_p^{1/p}$ admits a decomposition (1.1). However, more was proved in this case in [10] (see also [1]):

For any $f \in B_p^{1/p}$ there exists a decomposition (1.1) such that both v and g belong to $B_p^{1/p}$, and

$$\|v\|_{B_p^{1/p}(\mathbb{T})} + \|v\|_{L^\infty(\mathbb{T})} + \|g\,|_{\mathbb{T}}\|_{B_p^{1/p}(\mathbb{T})} \leqslant C\,\|f\|_{B_p^{1/p}(\mathbb{T})}. \tag{1.4}$$

2. FINITE BLASCHKE PRODUCTS AND THEIR LEVEL SETS

2.1. *Definitions*

Let

$$B(z) = \prod_{k=1}^{n} \frac{z_k - z}{1 - \bar{z}_k z}$$

be a Blaschke product of degree n with respect to the unit disc.

It has zeros z_k inside $\mathbb{D}$ and poles $1/\bar{z}_k$ outside $\mathbb{D}$. On $\mathbb{T}$ $|B(z)| = 1$.

Define the level set

$$E = \{z \in \mathbb{D} : \tfrac{1}{4} < |B(z)| < \tfrac{1}{2}\}$$

and its reflection with respect of $\mathbb{T}$

$$E^* = \left\{ z \in \mathbb{C} \backslash \mathbb{D} : \frac{1}{\bar{z}} \in E \right\} = \{ z \in \mathbb{C} \backslash \mathbb{D} : 2 < |B(z)| < 4 \}.$$

If B has a zero of order 2 or more at the origin, then the level set E^* lies in the annulus $\{1 < |z| < 2\}$.

Let $z \in E$. $|B| \leqslant 1$ in $\mathbb{D}$, and so [7, Chap. 1, Sect. 1.1] $|B| \leqslant \frac{4}{5}$ on the whole Whitney disc $\varDelta(z)$.

2.2. *Estimate of Hyperbolic Area of Level Set*

The following equality is well-known; we include its proof for the reader's sake.

LEMMA.

$$\int_{\mathbb{D}} |B'(z)|^2 \, dx \, dy = \pi n.$$

Proof. By the Green formula

$$\int_{\mathbb{D}} |B'(z)|^2 \, dx \, dy = \int_{\mathbb{D}} B'(x) \, \overline{B'(z)} \, dx \, dy = \frac{1}{2i} \int_{\mathbb{T}} B'(z) \, \overline{B(z)} \, dz$$

$$= \frac{1}{2i} \int_{\mathbb{T}} \frac{B'(z)}{B(z)} \, dz = \pi n. \quad \blacksquare$$

We use this result in the proof of the following estimate.

THEOREM 1.

$$\int_{\mathbb{D}} \left(\frac{1 - |B(z)|}{1 - |z|} \right)^2 dx \, dy \leqslant 8\pi(n+1).$$

Proof. The contribution of the disc $\{|z| < \frac{1}{2}\}$ to the integral is obviously less than π. As for the annulus $\{\frac{1}{2} < |z| < 1\}$, its contribution in polar coordinates is

$$\int_{1/2}^{1} \frac{r}{(1-r)^2} \, dr \int_0^{2\pi} (1 - |B(re^{i\theta})|)^2 \, d\theta$$

$$\leqslant \int_{1/2}^{1} \frac{r}{(1-r)^2} \, dr \int_0^{2\pi} |B(e^{i\theta}) - B(re^{i\theta})|^2 \, d\theta$$

$$\leqslant \int_0^{2\pi} d\theta \int_{1/2}^{1} \frac{r}{(1-r)^2} \, dr \left(\int_r^1 |B'(se^{i\theta})| \, ds \right)^2$$

By the well-known Hardy inequality [11, Appendix A.4] this is less than

$$4 \int_0^{2\pi} d\theta \int_{1/2}^{1} |B'(se^{i\theta})|^2 \, ds \leqslant 8 \int_{\mathbb{D}} |B'(z)|^2 \, dx \, dy = 8\pi n.$$

The last step follows from the Lemma. $\blacksquare$

The proven theorem leads us to some geometric properties of level sets of B.

COROLLARY 1.

$$\int_E \frac{dx\,dy}{(1-|z|)^2} \leqslant 32\pi(n+1). \tag{2.1}$$

The estimate (2.1) of the hyperbolic area of E may be written in another form.

COROLLARY 2. *The level set E may be covered by not more than Cn Whitney discs, with some absolute constant C.*

Proof. By the well-known Whitney construction [11, Chap. 6, Sect. 1] one can cover the whole $\mathbb{D}$ by Whitney discs with bounded multiplicity of intersections. Each Whitney disc intersecting E contributes a positive constant to the integral of Theorem 1 (see the last paragraph of Sect. 2.1). ∎

Consider for any point $e^{i\theta} \in \mathbb{T}$ the Luzin cone $\Gamma(e^{i\theta})$ and the maximal function for E

$$u(e^{i\theta}) = \max\left\{\frac{1}{1-|z|} : z \in E \cap \Gamma(e^{i\theta})\right\}. \tag{2.2}$$

COROLLARY 3.

$$\int_0^{2\pi} u(e^{i\theta})\,d\theta \leqslant Cn$$

with some absolute constant C.

Proof. Consider the covering of E by Whitney discs from Corollary 2. Fix a disc Δ and denote by $I(\Delta)$ the set of θ such that

$$\max\left\{\frac{1}{1-|z|} : z \in E \cap \Gamma(e^{i\theta})\right\}$$

is attained at some point $z \in \Delta$.

Then $E \cap \Gamma(e^{i\theta})$ is non-empty, and so $I(\Delta)$ is contained in an arc of length $C\,\mathrm{diam}(\Delta)$. Thus, the contribution of $I(\Delta)$ to $\int u\,d\theta$ is bounded by an absolute constant. ∎

2.3. *Pseudoanalytic Extension*

The Blaschke product B in $\mathbb{D}$ has an analytic extension $1/\overline{B(1/\bar{z})}$ outside $\mathbb{D}$. However, this extension has poles. Here we construct another extension, which is not analytic anymore, but has no singularity in the whole plane. The new extension is given by the formula

$$\widetilde{B}(z) = \frac{1}{\overline{B(1/\bar{z})}} \chi\{|B(1/\bar{z})|\}, \qquad |z| > 1. \tag{2.3}$$

Here χ is the cut-off function defined in Section 1.1.

The function $\widetilde{B}$ is not analytic, but it is C^∞ in the whole plane and $|\widetilde{B}| \leqslant 4$ everywhere. The Cauchy–Riemann derivative $\partial\widetilde{B}/\partial\bar{z}$ is supported on the set E^* and satisfies the estimate

$$\left|\frac{\partial \widetilde{B}}{\partial \bar{z}}\right| \leqslant C \frac{1}{|z|-1}, \qquad z \in E^*. \tag{2.4}$$

Indeed, due to the analyticity of the first factor in (2.3),

$$\begin{aligned}\left|\frac{\partial \widetilde{B}}{\partial \bar{z}}\right| &= \left|\frac{1}{\overline{B(1/\bar{z})}} \frac{\partial}{\partial \bar{z}} [\chi\{|B(1/\bar{z})|\}]\right| \\ &\leqslant 4 \cdot \max|\chi'| \cdot |B'(1/\bar{z})| \cdot \frac{1}{|z|^2} \\ &\leqslant C \cdot \frac{1}{1-|1/\bar{z}|} \cdot \frac{1}{|z|^2} \leqslant C \frac{1}{|z|-1}\end{aligned}$$

on E^*.

3. EXTENSION OF RATIONAL FUNCTIONS

3.1. *Construction*

Let r be a rational function of degree n with poles outside $\mathbb{D}$. We construct a pseudoanalytic extension of r to $\mathbb{C}\backslash\mathbb{D}$.

Let B be a Blaschke product of degree $n+2$ with the same poles as r and two additional poles at infinity. The ratio r/B is analytic in $\mathbb{C}\backslash\mathbb{D}$ and vanishes at infinity.

Fix any p, $1 \leqslant p \leqslant \infty$.

Let g be any function of the Hardy class $H^p(\mathbb{C}\backslash\mathbb{D})$. Put $f = r - g$. Then the function

$$h = \frac{f}{B} = \frac{r}{B} - \frac{g}{B}$$

is analytic in $\mathbb{C}\backslash\mathbb{D}$ and belongs to H^p.

On $\mathbb{T}$

$$|h| = |f|.$$

Define now an extension of r to $\mathbb{C}\backslash\mathbb{D}$ by the formula

$$\tilde{r} = h\widetilde{B} + g. \tag{3.1}$$

Evidently, $\tilde{r}$ is analytic outside the level set E^*. The latter lies in the annulus $\{1 < |z| < 2\}$ in our case.

On the set

$$U = \{z \colon |B(z)| < 2\}, \tag{3.2}$$

and so in some neighborhood of $\overline{\mathbb{D}}$, $\tilde{r}$ coincides with r.

On E^*, according to (2.4),

$$\left|\frac{\partial \tilde{r}}{\partial \bar{z}}\right| = \left|h(z)\,\frac{\partial \widetilde{B}}{\partial \bar{z}}\right| \leqslant C\,|h(z)|\,\frac{1}{|z|-1}. \tag{3.3}$$

In particular, if z belongs to an outer Luzin cone $\Gamma^*(e^{i\theta})$, then

$$\left|\frac{\partial \tilde{r}}{\partial \bar{z}}\right| \leqslant Ch^*(e^{i\theta})\,u(e^{i\theta}), \tag{3.4}$$

where

$$h^*(e^{i\theta}) = \sup\,\{|h(z)|, : z \in \Gamma^*(e^{i\theta})\}$$

is the non-tangential maximal function for h, and u is defined by (2.2).

On the circle $\{z \colon |z| = 2\}$

$$|\tilde{r}| \leqslant C(\|r\|_{L^1(\mathbb{T})} + \|g\|_{L^1(\mathbb{T})}),$$

where C is an absolute constant.

3.2. *Application*: *Bernstein Type Inequalities*

The construction of Section 3.1 allows us to give a new proof of the well-known Bernstein type inequalities for rational functions. These inequalities were proved first by V. V. Peller [9] in the BMO case and by A. A. Pekarskii [8] in the L^q case.

Let $q \geqslant 1$, $p > 0$, and

$$s = \frac{1}{p} - \frac{1}{q} > 0.$$

Consider the Besov class $B_p^s(\mathbb{T})$ (see Sect. 1.3).

THEOREM 2. *For any rational function r of degree n with poles outside $\mathbb{D}$*

$$\|r\|_{B^s_p(\mathbb{T})} \leqslant Cn^s \|r\|_{L^q(\mathbb{T})}.$$

Here C is a constant not depending on n.

Proof. Use the extension (3.1) with $g=0$. This means that $|h|=|r|$ on $\mathbb{T}$. According to (1.2), it suffices to check that

$$\int_{1<|z|<2} \sigma(z)^p \frac{dx\, dy}{(|z|-1)^{ps+1}} \leqslant Cn^{ps} \|r\|^p_{L^q(\mathbb{T})}, \tag{3.5}$$

where $\sigma(z)$ is defined by (1.3).

Set $z=(1+t)\, e^{i\theta}$, $t>0$. In our case, due to (3.4), $\sigma(z) \leqslant Ch^*(e^{i\theta})$. In the domain of integration $t>1/u(e^{i\theta})$ by definition (2.2) of u. Hence the above integral does not exceed

$$C \cdot \int_0^{2\pi} h^*(e^{i\theta})^p \, d\theta \int_{1/u(e^{i\theta})}^{\infty} \frac{dt}{t^{ps+1}} \leqslant C \cdot \int_{\mathbb{T}} h^{*p} u^{ps}.$$

By the Hölder inequality, the maximal theorem, and Corollary 3, the last integral is less than

$$\left(\int h^{*q}\right)^{p/q} \left(\int u\right)^{ps} \leqslant Cn^{ps} \|h\|^p_{H^q(\mathbb{C}\setminus\mathbb{D})} = Cn^{ps} \|r\|^p_{L^q(\mathbb{T})}. \quad \blacksquare$$

Only finite q occurs in Theorem 2. The following result is its counterpart for $q=\infty$.

THEOREM 3. *For any rational function r of degree n with poles outside $\mathbb{D}$ and any $p>0$*

$$\|r\|_{B^{1/p}_p(\mathbb{T})} \leqslant Cn^{1/p} \|r\|_{\mathrm{BMO}(\mathbb{T})}.$$

Proof. Apply the decomposition (1.1) to r and consider the extension (3.1) of r with this very g. One obtains, as above,

$$\sigma(z) \leqslant C \|h\|_{H^\infty} = C \|f\|_{L^\infty} \leqslant C \|r\|_{\mathrm{BMO}}.$$

Therefore, the integral (3.5) (where $s=1/p$ now) does not exceed

$$C \|r\|^p_{\mathrm{BMO}} \int_0^{2\pi} d\theta \int_{1/u(e^{i\theta})}^{\infty} \frac{dt}{t^2} \leqslant Cn \|r\|^p_{\mathrm{BMO}(\mathbb{T})}. \quad \blacksquare$$

Remark. It is well-known (see [3] for example) that the inverse theorems of the rational approximation theory follow from the Bernstein-type inequalities and some standard techniques of the interpolation space theory.

In particular, the inverse parts of Peller's and Pekarskii's approximation theorems [8, 9] are immediate corollaries of Theorems 2 and 3.

3.3. *Transfer to the Segment*

One can transfer the results of Section 3.2 from the disc $\mathbb{D}$ to the segment $I=[-1,1]$ of he real line using the Faber operator techniques [4, 5].

THEOREM 2′. *For any rational function r of degree n with poles outside I*

$$\|r\|_{B^s_p(I)} \leqslant Cn^s \|r\|_{L^q(I)}.$$

Here, as before, $p>0$, $q \geqslant 1$,

$$s=\frac{1}{q}-\frac{1}{p}>0,$$

and C is a constant not depending on n.

THEOREM 3′. *For any rational function r of degree n with poles outside I and any $p>0$*

$$\|r\|_{B^{1/p}_p(I)} \leqslant Cn^{1/p} \|r\|_{\mathrm{BMO}(I)}.$$

For the sake of simplicity we expose here only the proof of Theorem 3′.

Proof. Use the Faber operators constructed in [4, 5].
Let $f \in \mathrm{BMO(I)}$. Define the Faber transform Tf of f as

$$Tf(z)=\frac{1}{2\pi}\int_0^{2\pi} f(\cos t)\,\frac{e^{it}\,dt}{e^{it}-z}, \qquad z \in \mathbb{D}.$$

This is an analytic function in the unit disc. Since the change of variable $t \to \cos t$ preserves BMO space [7, Chap. 6, Corollary 1.3], we have

$$\|Tf\|_{\mathrm{BMOA}} \leqslant C\, \|f\|_{\mathrm{BMO(I)}}.$$

The operator T is one-to-one correspondence between BMO(I) and BMOA. For any rational function r with poles outside I its transform Tr is a rational function of the same degree with poles outside $\mathbb{D}$.

It is proved in [4, 5] that $f \in B_p^{1/p}(I)$ if and only if $Tf \in B_p^{1/p}(\mathbb{T})$ and

$$\|TF\|_{B_p^{1/p}(\mathbb{T})} \asymp \|f\|_{B_p^{1/p}(I)}.$$

Thus, the operator T is an isomorphism in the whole scale $B_p^{1/p}$.

Now for a rational r Theorem 3 asserts that

$$\|Tr\|_{B_p^{1/p}(\mathbb{T})} \leqslant Cn^{1/p} \|Tr\|_{\mathrm{BMOA}}.$$

The last estimate proves the theorem. ∎

Remark. Another Faber operator, constructed in [4, 5] as well, provides an isomorphism between $B_p^s(I)$ and the subspace of analytic functions in $B_p^s(\mathbb{T})$. For q fixed and

$$\frac{1}{q} = s + \frac{1}{p},$$

this isomorphism does not depend on p and is an isomorphism between $L^q(I)$ and the Hardy space $H^q(\mathbb{D})$. Therefore, one can repeat the proof above to prove Theorem 2′ too.

4. ESTIMATES IN AREA NORM

4.1. *An Estimate Outside the Disc*

Let r be a rational function of degree n with poles outside $\mathbb{D}$.

Let B be a Blaschke product of degree $n+2$ with the same poles as r and two additional poles at infinity. Consider the function

$$f(z) = \frac{r(z)}{B(z)}. \tag{4.1}$$

This is a new rational function, analytic in $\mathbb{C} \backslash \mathbb{D}$; $f(\infty) = 0$.

On the unit circle $|f|$ coincides with $|r|$. So, for example, all H^p norms of f and r (in $\mathbb{D}$ and $\mathbb{C} \backslash \mathbb{D}$, respectively) are the same. We intend to obtain a result of this kind for area L^p norms, when one cannot use boundary values.

THEOREM 4. *For any $p \geqslant 2$*

$$\left(\int_{\mathbb{C} \backslash \mathbb{D}} |f(z)|^p \, dx \, dy \right)^{1/p} \leqslant C_p n^{1/2} \left(\int_{\mathbb{D}} |r(z)|^p \, dx \, dy \right)^{1/p}, \tag{4.2}$$

where the constant C_p depends on p only.

Remark. The factor $n^{1/2}$ is sharp for $p=2$. One can check this on polynomials $r(z)=z^n$. The sharp exponent for $p>2$ is unknown.

Proof. It suffices to prove that for any continuous function φ of compact support in $\mathbb{C}\backslash\mathbb{D}$

$$\left|\int_{\mathbb{C}\backslash\mathbb{D}} f(\zeta)\,\varphi(\zeta)\,d\xi\,d\eta\right| \leqslant C_p n^{1/2}\,\|r\|_{L^p(\mathbb{D})}\,\|\varphi\|_{L^{p'}(\mathbb{C}\backslash\mathbb{D})}, \tag{4.3}$$

where $p'=p/(p-1)$ is the dual exponent.

Consider the function

$$g(z)=-\frac{1}{2\pi i}\int_{\mathbb{C}\backslash\mathbb{D}}\frac{\varphi(\zeta)}{\zeta-z}\,d\xi\,d\eta.$$

This function is analytic in $\overline{\mathbb{D}}$, and by the Green formula,

$$\begin{aligned}\int_{\mathbb{C}\backslash\mathbb{D}} f(\zeta)\,\varphi(\zeta)\,d\xi\,d\eta &= \int_{\mathbb{T}} f(z)\,g(z)\,dz\\ &= \int_{\mathbb{T}} r(z)\,g(z)\,\overline{B(z)}\,dz = 2i\int_{\mathbb{D}} rg\,\frac{\partial\overline{B}}{\partial\bar{z}}\,dx\,dy.\end{aligned}$$

In the case $p>2$ the Hardy–Littlewood–Sobolev fractional integration theorem [11, Chap. 5, Theorem 1 of Sect. 1] gives

$$\|g\|_{L^q(\mathbb{R}^2)} \leqslant C_p\,\|\varphi\|_{L^{p'}}, \tag{4.4}$$

where

$$\frac{1}{q}=\frac{1}{p'}-\frac{1}{2}=\frac{1}{2}-\frac{1}{p}.$$

Therefore, due to the Hölder inequality,

$$\begin{aligned}\left|\int_{\mathbb{C}\backslash\mathbb{D}} f(\zeta)\,\varphi(\zeta)\,d\xi\,d\eta\right| &\leqslant C_p\int_{\mathbb{D}}|r|\,|g|\,|B'|\,dx\,dy\\ &\leqslant C_p\left(\int_{\mathbb{D}}|r|^p\right)^{1/p}\left(\int_{\mathbb{D}}|g|^q\right)^{1/q}\left(\int_{\mathbb{D}}|B'|^2\right)^{1/2}\\ &\leqslant C_p n^{1/2}\,\|r\|_{L^p(\mathbb{D})}\,\|\varphi\|_{L^{p'}(\mathbb{C}\backslash\mathbb{D})}.\end{aligned}$$

The last inequality follows from (4.4) and the Lemma of Section 2.1.

Thus, the estimate (4.3) for the case $p>2$ is proved.

The Hardy–Littlewood–Sobolev theorem fails for $q=\infty$ [11, Chap. 5, Sect. 1.2]. The constant C_p in (4.4), and so in (4.2), tends to infinity as $p\to 2$. Nevertheless, there is another way to prove the theorem for $p=2$.

By a well-known result [11, Chap. 2, Theorem 3 of Sect. 4.2]

$$\|\nabla g\|_{L^2(\mathbb{R}^2)} \leqslant C\,\|\varphi\|_{L^2(\mathbb{R}^2)},$$

and, therefore [11, Chap. 6, Sect. 4.3], the following estimate holds:

$$\|g\,|_{\mathbb{T}}\|_{B_2^{1/2}(\mathbb{T})} \leqslant C\,\|\varphi\|_{L^2(\mathbb{R}^2)}.$$

According to Section 1.3, there exists a special decomposition (1.1) for g;

$$g\,|_{\mathbb{T}} = v + h\,|_{\mathbb{T}}, \qquad v\in L^\infty(\mathbb{T}), \quad h\in \mathrm{BMOA},$$

satisfying the estimate

$$\|v\|_{B_2^{1/2}(\mathbb{T})} + \|v\|_{L^\infty(\mathbb{T})} + \|h\,|_{\mathbb{T}}\|_{B_2^{1/2}(\mathbb{T})} \leqslant C\,\|g\,|_{\mathbb{T}}\|_{B_2^{1/2}(\mathbb{T})}.$$

Proving the key estimate (4.3) as before one obtains

$$\begin{aligned}\int_{\mathbb{C}\setminus\mathbb{D}} f(\zeta)\,\varphi(\zeta)\,d\xi\,d\eta &= \int_{\mathbb{T}} f(z)\,g(z)\,dz\\ &= \int_{\mathbb{T}} f(z)\,v(z)\,dz = \int_{\mathbb{T}} r(z)\,v(z)\,\overline{B(z)}\,dz\end{aligned}$$

(the contribution of h vanishes by analyticity).

Define the function V as the Poisson integral of v with respect to $\mathbb{D}$. This is a harmonic function in $\mathbb{D}$, bounded by the same constant as v, and [11, Chap. 5, Prop. 7′ and Sect. 5.1]

$$\|\nabla V\|_{L^2(\mathbb{D})} \leqslant C\,\|v\|_{B_2^{1/2}} \leqslant C\,\|\varphi\|_{L^2(\mathbb{R}^2)}.$$

Now we can finish the estimate as

$$\begin{aligned}&\int_{\mathbb{T}} r(z)\,v(z)\,\overline{B(z)}\,dz\\ &\qquad = 2i\int_{\mathbb{D}} r(z)\,\overline{B(z)}\,\frac{\partial V}{\partial\bar z}\,dx\,dy + 2i\int_{\mathbb{D}} r(z)\,V(z)\,\overline{B'(z)}\,dx\,dx. \qquad (3.5)\end{aligned}$$

However,

$$\left|\int_{\mathbb{D}} r\bar B\,\frac{\partial V}{\partial\bar z}\right| \leqslant \left(\int |r|^2\right)^{1/2}\left(\int |\nabla V|^2\right)^{1/2} \leqslant C\,\|r\|_{L^2(\mathbb{D})}\,\|\varphi\|_{L^2(\mathbb{R}^2)},$$

and

$$\left|\int_{\mathbb{D}} rV\overline{B'}\right| \leqslant \left(\int |r|^2\right)^{1/2} \|v\|_\infty \left(\int_{\mathbb{D}} |B'|^2\right)^{1/2} \leqslant Cn^{1/2} \|r\|_{L^2(\mathbb{D})} \|\varphi\|_{L^2(\mathbb{R}^2)}. \quad \blacksquare$$

The set $\mathbb{C}\backslash U$ (see (3.2)) is a "hyperbolic neighborhood" of the pole set of r.

COROLLARY 4. *For any* $p \geqslant 2$

$$\int_U |r|^p \leqslant C_p n^{p/2} \int_{\mathbb{D}} |r|^p.$$

4.2. *Application: An Inversion of the Sobolev Embedding Theorem*

The following embedding theorem for Sobolev classes is well-known [11, Chap. 5, Theorem 2 of Sect. 2.2]:

$$W_q^1(\mathbb{D}) \subset L^p(\mathbb{D}), \qquad 2 \leqslant p < \infty, \qquad \frac{1}{q} = \frac{1}{p} + \frac{1}{2}.$$

It turns out that this embedding admits a complete inversion for rational functions of a given degree.

THEOREM 5. *Let* r *be a rational function of degree* n *with poles outside* $\mathbb{D}$. *For any* p, $2 < p < \infty$,

$$\|r\|_{W_q^1(\mathbb{D})} \leqslant C_p n \, \|r\|_{L^p(\mathbb{D})}.$$

Remarks. (i) The estimate $\|r\|_{W_1^1} \leqslant C(n)\|r\|_{L^2}$ fails. A true analogue of the theorem in L^2 is not known yet.

(ii) Probably, the multiplier n is not sharp. We conjecture that the sharp exponent is 1/2, and the estimate must be true with the multiplier $n^{1/2}$.

Proof. Consider the extension $\tilde{B}$ of the Blaschke product B from Section 2.3 and define an extension of the function r by the formula

$$\tilde{r}(z) = f(z)\, \tilde{B}(z), \qquad z \in \mathbb{C}\backslash\mathbb{D}.$$

Here the function f is defined in (4.1).

The function $\tilde{r}$ is compactly supported and analytic outside the level set E^*. On E^*, due to (2.4),

$$\left|\frac{\partial \tilde{r}}{\partial \bar{z}}\right| = |f(z)| \left|\frac{\partial \tilde{B}}{\partial \bar{z}}\right| \leqslant C \frac{|f(z)|}{|z|-1}.$$

By the Cauchy–Green formula

$$r'(z) = -\frac{1}{\pi} \int_{E^*} \frac{\partial \tilde{r}}{\partial \bar{\zeta}} \frac{d\xi\, d\eta}{(\zeta - z)^2}, \qquad z \in \mathbb{D}.$$

According to Calderon–Zygmund estimate in the plane [11, Chap. 2, Theorem 3 of Sect. 4.2]

$$\|r'\|_{L^q(\mathbb{D})} \leqslant C_q \left\| \frac{\partial \tilde{r}}{\partial \bar{\zeta}} \right\|_{L^q(E^*)}.$$

Corollary 2 asserts that

$$E^* \subset \bigcup_{k=1}^{N} \Delta_k,$$

where Δ_k are Whitney discs and $N \leqslant Cn$. Therefore,

$$\begin{aligned} \int_{E^*} \left| \frac{\partial \tilde{r}}{\partial \bar{\zeta}} \right|^q &\leqslant C \int_{\cup \Delta_k} |f|^q \frac{1}{(|\zeta| - 1)^q} \\ &\leqslant C \left(\int_{\cup \Delta_k} |f|^p \right)^{q/p} \left(\int_{\cup \Delta_k} \frac{1}{(|\zeta| - 1)^{pq/(p-q)}} \right)^{1 - q/p}. \end{aligned}$$

Theorem 4 gives an estimate for the first factor. It does not exceed $Cn^{q/2} \|r\|^q_{L^p(\mathbb{D})}$.

In the second factor the exponent $pq/(p-q)$ equals 2, and the contribution of each Δ_k to the integral does not exceed an absolute constant. So the whole integral is less than CN, and we obtain the inequality

$$\left\| \frac{\partial \tilde{r}}{\partial \bar{\zeta}} \right\|_{L^q(E^*)} \leqslant C_p n \, \|r\|_{L^p(\mathbb{D})}. \quad \blacksquare$$

4.3. *A New Inverse Theorem of the Rational Approximation Theory*

Theorem 5 is an inequality of the Bernstein type for rational functions. As usual [3] it leads to the corresponding inverse theorem of the approximation theory.

In order to obtain such a result we need a slightly different version of the Bernstein inequality.

THEOREM 5′. *Let r be a rational function of degree n with poles outside $\mathbb{D}$. Let $2 < p < \infty$ and $0 < q < p/2$. Then*

$$\|r\|_{B^s_q(\mathbb{T})} \leqslant Cn^{1/2 + 1/q - 1/p} \, \|r\|_{L^p(\mathbb{D})},$$

where

$$s = \frac{1}{q} - \frac{2}{p}.$$

Proof. Consider the same extension $\tilde{r}$ of r as in the proof of Theorem 5 and apply the estimate (1.2). In this case the function σ of (1.3) is supported on the level set $\{z: |B(z)| \geqslant 5/4\}$ due to the remark at the end of Section 2.1. So the domain of integration in (1.2) is contained in a union of not more than Cn Whitney's discs Δ_k. Now by the Hölder inequality

$$\begin{aligned} &\int_{1<|z|<2} \sigma(z)^q \frac{dx\,dy}{(|z|-1)^{qs+1}} \\ &\leqslant \left(\int \sigma(z)^p \,dx\,dy \right)^{q/p} \left(\int_{\cup \Delta_k} \frac{dx\,dy}{(|z|-1)^{(qs+1)p/(p-q)}} \right)^{1-q/p} \\ &\leqslant Cn^{1-q/p} \left(\int \sigma^p \right)^{q/p}, \end{aligned}$$

because

$$(qs+1)\frac{p}{p-q} = 2.$$

However, due to the mean value theorem and to (4.2),

$$\|\sigma\|_{L^p(\mathbb{C}\setminus\mathbb{D})} \leqslant C\,\|f\|_{L^p(\mathbb{C}\setminus\mathbb{D})} \leqslant Cn^{1/2}\,\|r\|_{L^p(\mathbb{D})}.$$

Thus,

$$\|r\|_{B_q^s(\mathbb{T})} \leqslant Cn^{1/2+1/q-1/p}\,\|r\|_{L^p(\mathbb{D})}. \quad \blacksquare$$

According to the general theory of approximation space [3, Corollary 1 on p. 129] Theorem 5′ implies the following inverse theorem of the rational approximation theory.

Let $2 < p < \infty$. For any function f, analytic in $\mathbb{D}$, such that $f \in L^p(\mathbb{D})$, define its best rational approximation in the L^p norm,

$$R_n(f)_p = \inf \|f - r\|_{L^p(\mathbb{D})}, \qquad n = 1, 2, ...,$$

where inf is taken over all rational functions r of degree n with poles outside $\mathbb{D}$.

THEOREM 6. *For any* q, $0<q<p/2$, *if* $f\in L^p(\mathbb{D})$ *is analytic and*

$$\sum n^{q(1/2-1/p)} R_n(f)_p^q < \infty,$$

then

$$f \in B_q^{1/q-2/p}(\mathbb{T}).$$

Proof. Choose two different exponents q_1 and q_2, $0<q_1<q<q_2<p/2$. Then by Theorem 5′

$$\|r\|_{B_{q_i}^{s_i}} \leqslant C_i n^{\alpha_i} \|r\|_{L^p(\mathbb{D})}, \qquad i=1, 2.$$

Here

$$s_i = \frac{1}{q_i} - \frac{2}{p}, \qquad \alpha_i = \frac{1}{2} + \frac{1}{q_i} - \frac{1}{p},$$

and r is an arbitrary rational function of degree n with poles outside $\mathbb{D}$.

Therefore Corollary 1 on page 129 in [3] asserts that for any t, $0<t<1$, if

$$\sum_{n=1}^{\infty} \frac{1}{n} [n^{\alpha_t} R_n(f)]^q < \infty,$$

then

$$f \in [B_{q_1}^{s_1}, B_{q_2}^{s_2}]_{tq},$$

where

$$\alpha_t = (1-t)\,\alpha_1 + t\alpha_2,$$

and $[X_1, X_2]_{tq}$ denotes the corresponding real interpolation space between (quasi)-Banach spaces X_1 and X_2.

Choosing t so that

$$\frac{1}{q} = \frac{1-t}{q_1} + \frac{t}{q_2},$$

and taking into account that [3]

$$[B_{q_1}^{s_1}, B_{q_2}^{s_2}]_{tq} = B_q^s,$$

where

$$s = (1-t)\, s_1 + t s_2 = \frac{1}{q} - \frac{2}{p},$$

one obtains the desired result. ▮

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